A Brief Introduction to Characters and Representation Theory

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Overview

Material studied: *Linear Representations of Finite Groups* by Jean-Pierre Serre

What is Representation Theory?

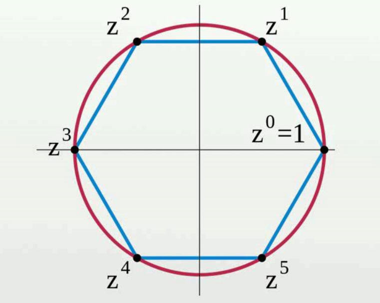
Representation theory is the study of algebraic structures by representing the structure's elements as linear transformations of vector spaces.

This makes abstract structures more concrete by describing the structure in terms of matrices and their algebraic operations as matrix operations.

Structures Studied

Structures studied this way include:

- Groups
- Associative Algebras
- Lie Algebras



Finite Groups

- Study group actions on structures.
  - especially operations of groups on vector spaces; other actions are group action on other groups or sets.

Group elements are represented by invertible matrices such that the group operation is matrix multiplication.

Linear Representations

Let V be a K-vector space and G a finite group. The linear representation of G is a group homomorphism $\rho : G \to GL(V)$.

So, a linear representation is a map $\rho : \ G \to GL(V) \ s.t. \ \rho(st) = \rho(s)\rho(t) \ \forall s, t \in G$.

Importance of Linear Representations

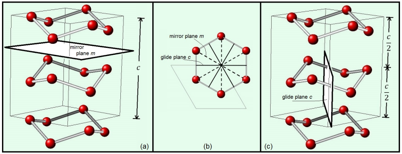
Linear representations allow us to state group theoretic problems in terms of linear algebra.

Linear algebra is well understood; reduces complexity of problems.

Applications of Linear Representations

Applications in the study of geometric structures and in the physical sciences.

- Space Groups
  - Symmetry groups of the configuration of space.



- Lattice Point Groups
  - Lattice groups define the geometric configuration of crystal structures in materials science and crystallography.

Characters

The character of a group representation is a function on the group that associates the trace of each group element's matrix to the corresponding group element.

Characters contain all of the essential information of the representation in a more condensed form.

Trace Review

Note: Trace is the sum of the diagonal entries of the matrix.

$$Tr(X) = Tr(\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix})$$

$$Tr(X) = x_{11} + x_{22} + \dots + x_{mn} = \sum_{i=1}^{n} x_{ii}$$

Characters:

For a representation $\rho : G \to GL(V)$ of a group G on V. the character of ρ is the function $\chi_\rho : G \to F$ given by $\chi_\rho(g) = Tr(\rho(g))$.

Where F is a field that the finite-dimensional vector space V is over.

Orthogonality of Characters

The space of complex-valued class functions of a finite group G has an inner product given by:

$$\{\alpha, \beta\} := \frac{1}{G} \sum_{g \in G} \alpha(g)\beta\bar{(}g) \tag{1}$$

From this inner product, the irreducible characters form an orthonormal basis for the space of class functions and an orthogonality relation for the rows of the character table. Similarly an orthogonality relation is established for the columns of the character table.

Consequences of Orthogonality:

Consequences of Orthogonality:

- An unknown character can be decomposed as a linear combination of irreducible characters.
- The complete character table can be constructed when only a few irreducible characters are known.
- The order of the group can be found.

Character Properties

- A character χ_ρ is irreducible if ρ is irreducible.
- One can read the dimension of the vector space directly from the character.
- Characters are class functions; take a constant value on a given conjugacy.
- The number of irreducible characters of G is equal to the number of conjugacy classes of G.
- The set of irreducible characters of a given group G into a field K form a basis of the K-vector space of all class functions $G \to K$.

Examples of Characters

Note: the elements of any group can be partitioned in to conjugacy classes; classes corresponding to the same conjugate element.

$$Cl(a) = \{b \in G | \exists g \in G \text{ with } b = gag^{-1}\} \qquad (2)$$

From the definition it follows that Abelian groups have a conjugacy class corresponding to each element.

Examples: Generalized Cyclic Group $\mathbb{Z}_n$

Note: All of $\mathbb{Z}_n$'s irreducible characters are linear.

$\mathbb{Z}_n$ is an additive group where $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}\}$

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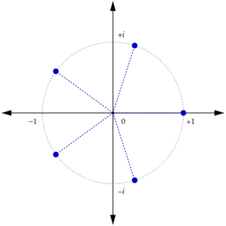
Let $\omega_n = e^{\frac{2\pi i}{n}}$ be a primitive n root of unity.
(any complex number that gives 1 when raised to a positive integer power)



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$|\mathrm{Irr}(\mathbb{Z}_n)| = n$.

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Let $\chi_0, \chi_1, \chi_2, \ldots, \chi_{n-1}$ be the n irreducible characters of $\mathbb{Z}_n$ then $\chi_m(\bar{j}) = \omega_n^{jm}$
where $0 \leq j \leq n-1$ and $0 \leq m \leq n-1$.

Character Table for $\mathbb{Z}_n$

g_i	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\dots$	$\overline{n-1}$
$\lvert Cl \rvert$	1	1	1	$\dots$	1
χ_0	1	1	1	$\dots$	1
χ_1	1	ω_n	ω_n^2	$\dots$	ω_n^{n-1}
χ_2	1	ω_n^2	ω_n^4	$\dots$	$\omega_n^{2(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
χ_{n-1}	1	ω_n^{n-1}	$\omega_n^{2(n-1)}$	$\dots$	$\omega_n^{(n-1)(n-1)}$

Cyclic Group $\mathbb{Z}_6$

We can find the character table for $\mathbb{Z}_6$ fairly easily.
$\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$.

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There are 6 conjugacy classes:
$\{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}, \{\bar{4}\}, \{\bar{5}\}$.

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There are 6 conjugacy classes: $\{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}, \{\bar{4}\}, \{\bar{5}\}$.

Let $\omega_6 = e^{\frac{2\pi i}{6}}$ be a primitive 6 root of unity; then $\chi_m(\bar{j}) = \omega_6^{jm}$.

Where $0 \leq j \leq 5$ and $0 \leq m \leq 5$.

Character Table: Cyclic Group $\mathbb{Z}_6$

$\omega_6^6 = 1$ due to cycle.

g_i	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\lvert Cl \rvert$	1	1	1	1	1	1
χ_0	1	1	1	1	1	1
χ_1	1	ω_6	ω_6^2	ω_6^3	ω_6^4	ω_6^5
χ_2	1	ω_6^2	ω_6^4	1	ω_6^2	ω_6^4
χ_3	1	ω_6^3	1	ω_6^3	1	ω_6^3
χ_4	1	ω_6^4	ω_6^2	1	ω_6^4	ω_6^2
χ_5	1	ω_6^5	ω_6^4	ω_6^3	ω_6^2	ω_6^1

Computing the Roots of Unity

We know that $\omega_6^6 = 1$.
Calculating the rest from $\omega_6 = e^{\frac{2\pi i}{6}}$:

$$\omega_6^1 = \frac{1}{2} + \frac{\sqrt{3}}{2} i,$$

$$\omega_6^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i,$$

$$\omega_6^3 = -1,$$

$$\omega_6^4 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i,$$

$$\omega_6^5 = \frac{1}{2} - \frac{\sqrt{3}}{2} i.$$

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